

Common Fixed Point Theorem of New Type of Contraction in Complete Metric Space.

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Abstract:-In this paper we define new type of contraction and Prove common fixed point theorem for the sequence of mappings of this new type of contraction in Complete Metric Space.

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1. Introduction

Metric fixed Point theory has tremendous applications in many Branches of Science and Engineering field. Metric fixed point theory developed after the Polish Mathematician S.Banach [1] proved Banach Contraction principle, which states that a contraction map on a complete Metric Space has a unique fixed point. Now Fixed point theory has a vast literature. In 1968 R.Kanan [2] proved a fixed point theorem for self map. In 1971 Chatterjea [3] proved a fixed point theorem for a self map which is a modification of Kanan map. In 1980 Khan M.S. [4] proved some fixed point theorems in Metric and Banach Space. Afterward Delbosko [5] defined set of all continuous functions $g : R_+^3 \rightarrow R_+$ which satisfying some properties and proved some fixed point results .In 1994AndrianConstantin [6] proved common fixed point theorem for two pair of weakly commuting maps. Now In this paper we define a new contraction and prove common fixed point theorems for sequence of maps.

2. Preliminaries

Definition 2.1:-Let X be a non-empty set . A mapping $d : X \times X \rightarrow R$ is said to be a Metric or a distance function if it satisfies following conditions.

1. $d(x, y)$ is non-negative.
2. $d(x, y) = 0$ if and only if x and y coincides i.e. $x=y$.
3. $d(x, y) = d(y, x)$ (Symmetry)
4. $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality)

Then the function d is referred to as metric on X . And (X,d) or simply X is said to as Metric space.

Definition 2.2:- A Metric space (X,d) is said to be a complete Metric space if every Cauchy sequence in X converges to a point of X .

Definition 2.3:- If (X,d) be a complete Metric space and a function $F : X \rightarrow X$ is said to be a contraction map if

$$d(F(x), F(y)) \leq \beta d(x, y)$$

For all $x, y \in X$ and for $0 < \beta < 1$

Definition 2.4:- Let $F : X \rightarrow X$, then $x \in X$ is said to be a fixed point of F if $F(x) = x$

Definition 2.5:-Let X be a Metric space and if F_1 and F_2 be any two maps. An element $a \in X$ is said to be a common fixed point of F_1 and F_2 if $F_1(a) = F_2(a)$

For ex:- If $F_1(x) = \sin(x)$ and $F_2(x) = \tan(x)$

Then 0 is the common fixed point F_1 and F_2 Since $F_1(0) = \sin(0)$ and $F_2(0) = \tan(0) = 0$

Theorem 2.1 (Kanan fixed point theorem in complete metric space) [2]

Let X be a Complete metric Space and $F : X \rightarrow X$ is a mapping such that,

$$d(Fx, Fy) \leq \beta [d(x, Fx) + d(y, Fy)] \text{ for all } x, y \in X$$

Where $\beta \in]0, 1[$ Then F has a unique fixed point.

Theorem 2.2[3] A mapping $F : X \rightarrow X$ where (X, d) is a Metric space is said to be C-Contraction if there is

a some β s.t. $0 < \beta < \frac{1}{2}$ s.t. the following inequality holds

$$d(F_x, F_y) \leq \beta (d(x, F_y) + d(y, F_x))$$

If (X, d) be a complete Metricspace, then any C-contraction on X has a unique fixed point.

In 1981 Delbosco [5] defined the set G of all continuous mappings $\alpha : [0, \infty)^3 \rightarrow [0, \infty)$ which satisfies the following conditions

(i) $\alpha(1, 1, 1) = k < 1$

(ii) Let $a, b \geq 0$ be such that either $a \leq k(a, b, b)$ or $a \leq k(b, b, a)$ or $a \leq k(b, a, b)$. Then $a \leq kb$.

And Delbosco proved that for $P : X \rightarrow X, Q : X \rightarrow X$ on a Complete metric space (X, d)

Which satisfies the condition,

$$d(Pa, Qb) \leq \alpha(d(a, b), d(a, Qb), d(b, Qb)) \tag{1}$$

For all $a, b \in X$, where $\alpha \in G$. Then P and Q have a unique common fixed point. Then in 1994 Adrian Constantin proved following result of common fixed point theorem.

Let P and Q be two self maps of a metric space (X, d) which satisfies following conditions

(i) $d(Pa, Qb) \leq \alpha(d(a, b), d(a, Pa), d(b, Qb))$

for all $a, b \in X$ and $\alpha \in G$

(ii) There is a point $v \in X$ so that P is continuous at v and Q is continuous at Pv ,

(iii) There exists a point $a \in X$ s.t. the sequence $\{(P \circ S)^n(a)\} = \{(Q \circ P)^n\}$ has a subsequence

$\{(QP)^n(a)\}$ converging to v . Then $v = Pv$ is the unique fixed point of P and Q .

And Delbosco proved common fixed point theorem for pair of two weakly commuting mappings also

Theorem 2.3[5] If P and R be weakly commuting mappings and if Q and S be weakly commuting self mappings of a complete metric space (X, d) into itself which satisfies the following conditions

$$d(Px, Qy) \leq \alpha(d(Rx, Sy), d(Rx, Px), d(Sy, Qy)) \text{ for all } x, y \in X$$

for $\alpha \in G$. If the range of R contains the range of Q and the range of S contains range of P , and if one of P, Q, R and S is continuous, then P, Q, R and S have a unique common fixed point z .

we modify Delboscos inequality for two maps given by equation (1) and prove some common fixed point theorems.

First we introduce β Let Ebe the set of all functions $\beta : [0, \infty)^3 \rightarrow [0, \infty)$ which satisfies

- (i) β is continuous on the set $[0, \infty)^3$ (with respect to Euclidean metric on $[0, \infty)^3$)
- (ii) $x \leq ky$ for some k s.t. $0 \leq k < 1$ whenever $x \leq \beta(y, y, x)$ or $x \leq \beta(y, x, y)$
or $x \leq \beta(x, y, y)$ for all $x, y \in [0, \infty)$
- (iii) $\beta(x, y, y) = 0$ iff $x=y=0$

Definition 2.6:- A mapping F on a metric space X into itself is said to be New contraction if it satisfies the following condition.

$$d(Fa, Fb) \leq \beta(d(a, b), d(a, Ta), d(b, Fb))$$

for every $a, b \in X$ and some $\beta \in E$.

Example 2.1 :- A mapping $F : X \rightarrow X$ defined by

$$d(Fa, Fb) \leq \alpha \max\{d(Fa, a) + d(Fb, b), d(Fb, b) + d(a, b), d(Fa, a) + d(a, b)\}$$

for all a, b in X and some $0 \leq \alpha < \frac{1}{2}$ is New type contraction.

The map $\beta : R_+^3 \rightarrow R_+$ is defined as

$$\beta(u, v, w) = \alpha \max\{u + v, v + w, u + w\}$$

for every $u, v, w \in R_+$, where α is s.t. $0 \leq \alpha < \frac{1}{2}$. Then as $\beta \in E$.

Clearly β is continuous. Also for $u \leq \beta(u, v, v) = \alpha \max\{u + v, v + u, v + v\}$

There are two possibilities

Case (i) If $\max\{u + v, v + u, v + v\} = u + v$

\therefore In this case

$$u \leq \frac{\alpha}{1 - \alpha} \leq kv, \text{ where } k = \frac{\alpha}{1 - \alpha} \text{ in } [0, 1)$$

Case (ii) If $\max\{u + v, v + u, v + v\} = v + v$

Case(ii) If $\therefore u \leq kv$ where $k=2\alpha$, for $0 \leq \alpha < 1$

\therefore we have for $u \leq \beta(v, u, v)$ or $u \leq \beta(v, v, u)$ we obtain $u \leq kv$ for k in $0 \leq \alpha < 1$.

$$\begin{aligned} d(Fa, Fb) &\leq \alpha \max\{d(Fa, a) + d(Fb, b), d(Fb, b) + d(a, b), d(Fa, a) + d(a, b)\} \\ &= \beta(d(a, b), d(Fa, a), d(Fb, b)) \end{aligned}$$

\therefore by definition F is new type contraction.

3 Main Result

Now we prove a fixed point result for new type contraction.

Theorem 3.1 :- If (X, d) be a Complete Metric space and if F be mapping on X into itself which satisfies

$$d(Fa, Fb) \leq \beta(d(a, b), d(a, Ta), d(b, Fb)) \tag{3.1}$$

for all $a, b \in X$ and some $\beta \in E$.

F has a Unique fixed point in X .

Proof :- Let a_0 be an arbitrary point in X . We construct a sequence $\{a_n\}$ in X as

$$a_{n+1} = Fa_n, a_1 = Fa_0, a_2 = Fa_1, \dots, a_{n+1} = Fa_n \text{ i.e. } a_n = F^n a_0$$

Given F satisfies (3.1)

$$\begin{aligned}
 d(a_n, a_{n+1}) &= d(Fa_{n-1}, Fa_n) \\
 &\leq \beta(d(a_{n-1}, a_n), d(Fa_{n-1}, a_{n-1}), d(Fa_n, a_n)) \\
 \text{Consider,} \quad &\leq \beta(d(a_{n-1}, a_n), d(a_n, a_{n-1}), d(a_{n+1}, a_n)) \\
 &\leq kd(a_{n-1}, a_n) \tag{3.2}
 \end{aligned}$$

Similarly,

$$d(a_{n-1}, a_n) \leq kd(a_{n-2}, a_{n-1})$$

∴ (3.2) gives

$$d(a_n, a_{n+1}) \leq k^2 d(a_{n-2}, a_{n-1})$$

$$\begin{aligned}
 &\cdot \quad \cdot \\
 &\cdot \quad \cdot \\
 &\cdot \quad \cdot
 \end{aligned}$$

$$d(a_n, a_{n+1}) \leq k^n d(a_0, a_1) \tag{3.3}$$

$$\text{for } 0 \leq k < 1$$

Letting $n \rightarrow \infty$ we have $\{a_n\}$ is a Cauchy sequence in X. And as X is complete.

∴ $\{a_n\}$ converges to a point in X. Let $\{a_n\}$ converges to $u \in X$

∴ for $a=u$ and $b=a_n$

inequality (3.1) gives

$$\begin{aligned}
 d(Fa, a_{n+1}) &= d(Fu, Fa_n) \\
 &\leq \beta(d(u, a_n), d(u, Fu), d(a_n, Fa_n)) \\
 &= \beta(d(u, a_n), d(u, Fu), d(a_n, a_{n+1}))
 \end{aligned}$$

Taking limit as $n \rightarrow \infty$ and since given β is continuous we have

$$d(Fu, u) \leq \beta(d(u, u), d(u, Fu), d(u, u))$$

$$\therefore d(Fu, u) \leq k \cdot 0 = 0$$

$$\text{Thus } d(Fu, u) = 0$$

this gives $Fu = u$

∴ u is a fixed point of F.

Uniqueness: - Now if possible suppose x be another fixed point of F

$$\therefore Fx = x$$

Now we put $a=x$ and $b=u$ in inequality (3.1), we have

Consider,

$$\begin{aligned}
 d(x, u) &= d(Tx, u) \\
 &\leq \beta(d(x, u), d(Fx, x), d(Fu, u)) \\
 &\leq \beta(d(x, u), d(w, w), d(u, u)) \\
 &\leq \beta(d(x, u), 0, 0)
 \end{aligned}$$

$$\therefore d(x, u) \leq k \cdot 0$$

$$\therefore d(x, u) = 0$$

This gives $x=u$.

Now we prove another result of common fixed point theorem for sequence of mappings of new type of contraction.

Theorem 3.2 Let (X, d) be a complete Metric space. And if $\beta \in E$ and the sequence $\{F_n\}_{n=1}^{\infty}$ of maps on X into itself is s.t.

$$d(F_i a, F_j b) \leq \beta(d(a, b), d(F_i a, a), d(F_j b, b)) \quad (3.4)$$

for all $a, b \in X$. Then sequence $\{F_n\}_{n=1}^{\infty}$ has a unique common fixed point in X .

Proof: -We construct a sequence $\{a_n\}$ of points of X s.t. for some fixed a_0 in X . For each n in N

We define $a_n = F_n a_{n-1}$, as $\beta \in E$, from inequality (3.4)

Consider,

$$\begin{aligned} d(a_1, a_2) &= d(F_1 a_0, F_2 a_1) \\ &\leq \beta(d(a_0, a_1), d(a_0, F_1 a_0), d(a_1, F_2 a_1)) \\ &= \beta(d(a_0, a_1), d(a_0, a_1), d(a_1, a_2)) \\ &\leq k d(a_0, a_1) \end{aligned} \quad (a)$$

for $0 \leq k < 1$.

similarly for $x_2, x_3 \in X$

$$\begin{aligned} d(a_2, a_3) &= d(F_2 a_1, F_3 a_2) \\ &\leq \beta(d(a_1, a_2), d(a_1, F_2 a_1), d(a_2, F_3 a_2)) \\ &= \beta(d(a_1, a_2), d(a_1, a_2), d(a_2, a_3)) \\ &\leq k d(a_1, a_2) \end{aligned} \quad (b)$$

from (a) and (b) we have

$$\therefore d(a_2, a_3) \leq k^2 d(a_0, a_1)$$

In general, We have

$$d(a_n, a_{n+1}) \leq k^n d(a_0, a_1)$$

For $0 \leq k < 1$. $\therefore \{a_n\}$ Is a Cauchy Sequence in X . Since X is complete. \therefore it converges to $u \in X$ consider,

$$\begin{aligned} d(u, F_n u) &\leq d(u, a_{m+1}) + d(a_{m+1}, F_n u) \\ &= d(u, a_{m+1}) + d(F_{m+1} a_m, F_n u) \\ &\leq d(u, a_{m+1}) + \beta(d(a_m, u), d(F_{m+1} a_m, a_m), d(F_n u, u)) \quad (\text{since by (3.4)}) \\ &\leq d(u, a_{m+1}) + \beta(d(a_m, u), d(a_{m+1}, a_m), d(F_n u, u)) \end{aligned}$$

For all $m, n \in N$. Letting $m \rightarrow \infty$ then above relation gives

$$\begin{aligned} d(u, F_n u) &\leq d(u, u) + \beta(d(u, u), d(u, u), d(F_n u, u)) \\ &\leq \beta(0, 0, d(F_n u, u)) \\ &\leq 0 \end{aligned}$$

$$\therefore d(u, F_n u) = 0$$

$$\therefore F_n u = u, \text{ for all } n \text{ in } N$$

This gives u is the common fixed point of sequence of maps $\{F_n\}$.

To prove Uniqueness of fixed point, If possible suppose w in X be another fixed point of F_n

$$\therefore F_n w = w$$

Now consider

$$\begin{aligned}d(u, w) &= d(F_1 u, F_1 w) \\ &\leq \beta(d(u, w), d(F_1 u, u), d(w, F_1 w)) \quad (\text{by (3.4)}) \\ &= \beta(d(u, w), d(u, u), d(w, w)) \\ &= \beta(d(u, w), 0, 0) \\ &\leq k \cdot 0\end{aligned}$$

$\therefore d(u, w) = 0$ This gives $u=w$. i.e. the common fixed point of $\{F_n\}$ is unique.

Conclusion: - Thus we have defined a new type of contraction and proved one result of fixed point and one result of common fixed point of sequence of mappings.

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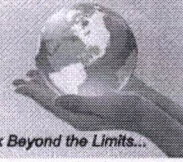
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