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On Results of Common Fixed Points of Compatible maps in Generalized metric Space

Research Article

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Abstract: In this paper we Prove some Common Fixed point theorems of compatible maps in G-metric Space.

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1. Introduction

The Fixed point Theory has wide applications in Many Branches of Science and Engineering field, S.Banach [1] obtained a well known theorem for a contraction mapping in a Complete metric space.Which states that,"A contraction has a Unique fixed point theorem in a Complete metric space." After that many authors proved fixed point theorems for mappings satisfying certain contraction conditions. Fixed point theory developed in three directions,Generalizations of contractive conditions,existence and if possible uniqueness of fixed points.The scope of metric fixed point theory becomes large with developing new metric space structures.And various fixed point results were obtained in new spaces by many authors.

In 1960 Gahler [4] introduced a new structure of 2-metric space and he claimed that every 2-metric space is the standard metric space.But some authors showed that there is relation between these two mappings. Also Ha et.al. [5] show that a 2-metric need not be continuous function of its variables,but ordinary metric is continuous.

In 1992 Dhage B.C. [3] introduced a new generalized notion of metric called as D-metric. In 2003 Z. Mustafa and Brailley Sims [8] proved that most of the fundamental topological properties claimed in D-metric space are not correct also the D-metric is not continuous of its variables. In 2006 Z.Mustafa and Brailley Sims [9] introduced new generalized metric called as generalized metric or G-metric space and several fixed point results are proved by Mustafa and Sims and by many authors. Sesa [10] introduced a concept of a weakly commuting mappings and obtained some common fixed point theorems in complete metric space. In 1986 Jungck [6] defined compatible mapping and proved some common fixed theorems in complete metric space. Chaudhry et al. [2] introduced the concept of compatible maps in G-metric space. Now here we prove common fixed point theorems of pair of self maps which are compatible in G-metric space.

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Regards,
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2. Preliminaries

Definition 2.1 (G-metric Space [9]). *Let X be a non empty set and $G : X^3 \to R^+$ which satisfies the following conditions*

(1). $G(a,b,c) = 0$ if $a = b = c$ i.e. every a, b, c in X coincides.

(2). $G(a,a,b) > 0$ for every $a,b,c \in X$ s.t. $a \neq b$

(3). $G(a,a,b) \leq G(a,b,c)$, $\forall$ $a,b,c \in X$ s.t. $c \neq b$

(4). $G(a,b,c) = G(b,a,c) = G(c,b,a) = \dots$ (symmetrical in all three variables)

(5). $G(a,b,c) \leq G(a,x,x) + G(x,b,c)$, for all a, b, c, x in X (rectangle inequality)

Then the function G is said to be generalized metric or simply G-metric on X and the pair (X,G) is said to be G-metric space.

Example 2.2 ([9]). *Let $G : X^3 \to R^+$ s.t. $G(a,b,c)$ = perimeter of the triangle with vertices at a,b,c in R^2, also by taking p in the interior of the triangle then rectangle inequality is satisfied and the function G is a G-metric on X.*

Remark 2.3. *G-metric space is the generalization of the ordinary metric space that is every G-metric space is (X,G) defines ordinary metric space (X, d_G) by $d_G(a,b) = G(a,b,b) + G(a,a,b)$.*

Example 2.4. *Let (X,d) be the usual metric space. Then the function $G : X^3 \to R^+$ defined by $G(a,b,c) = \max\{d(a,b), d(b,c), d(c,a)\}$ for all $a,b,c \in X$ is a G-metric space.*

Definition 2.5. *A G-metric space (X,G) is said to be symmetric if $G(a,b,b) = G(a,a,b)$ for all $a,b \in X$ and if $G(a,b,b) \neq G(a,a,b)$ then G is said to be non symmetric G-metric space.*

Example 2.6. *Let $X = \{x,y\}$ and $G : X^3 \to R^+$ defined by $G(x,x,x) = G(y,y,y) = 0$, $G(x,x,y) = 1$, $G(x,y,y) = 2$ and extend G to all of X^3 by symmetry in the variables. Then X is a G-metric space but It is non symmetric. since $G(x,x,y) \neq G(x,y,y)$.*

Definition 2.7. *Let (X,G) be a G-metric space, Let $\{a_n\}$ be a sequence of elements in X. The sequence $\{a_n\}$ is said to be G-convergent to a if $\lim_{m,n\to\infty} G(a,a_n,a_m) = 0$ i.e for every $\epsilon > 0$ there is N s.t. $G(a,a_n,a_m) < \epsilon$ for all $m,n \geq N$ It is denoted as $a_n \to a$ or $\lim_{n\to\infty} a_n = a$.*

Proposition 2.8 ([9]). *If (X,G) be a G-metric space. Then the following are equivalent*

(1). $\{a_n\}$ is G-convergent to a.

(2). $G(a_n,a_n,a) \to 0$ as $n \to \infty$

(3). $G(a_n,a,a) \to 0$ as $n \to \infty$

(4). $G(a_m,a_n,a) \to 0$ as $m,n \to \infty$

Definition 2.9. *Let (X,G) be a G-metric space a sequence $\{a_n\}$ is called G-Cauchy if, for each $\epsilon > 0$ there is an $N \in I^+$ (set of positive integers) s.t. $G(a_n,a_m,a_l) < \epsilon$ for all $n,m,l \geq N$.*

Proposition 2.10. *Let (X,G) be a G-metric space then the function G(a,b,c) is said to be jointly continuous in all three of its variables.*

Definition 2.11 ([6]). *Let f and g be two self maps on a metric space (X,d). The mappings f and g are said to be compatible if* $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ *for some* $z \in X$.

Proposition 2.12 ([9]). *Let (X,G) be a G-metric space. Then, for any a,b,c,x in X it gives that*

(1). *if* $G(a,b,c) = 0$ *then* $a = b = c$

(2). $G(a,b,c) \leq G(a,a,b) + G(a,a,c)$

(3). $G(a,b,b) \leq 2G(b,a,a)$

(4). $G(a,b,c) \leq G(a,x,c) + G(x,b,c)$

(5). $G(a,b,c) \leq \frac{2}{3}(G(a,x,x) + G(b,x,x) + G(c,x,x))$

Now we begin with the contraction mapping in G-metric space

Definition 2.13. *Let (X,G) be a G-metric space and* $T : X \to X$ *be a self mapping on X. Now T is said to be a contraction if* $G(Ta,Tb,Tc) \leq \alpha G(a,b,c)$ *for all* $a,b,c \in X$ *where* $0 \leq \alpha < 1$.

Definition 2.14 ([2]). *Let f and g be two self mappings on a G-metric space (X,G). Then mappings f and g are said to be compatible if* $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$, *whenever* $\{x_n\}$ *is a sequence in X s.t.* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ *for some* $z \epsilon X$ *for some z in X.*

Example 2.15. *Let X=[-1,1] and* $G : X^3 \to R^+$ *be defined as follows* $G(a,b,c) = |a-b| + |b-c| + |c-a|$ *for all a,b,c* ϵX. *Then (X,G) be a G-metric space. Let us define fa=a and ga=*$\frac{a}{4}$. *Let* $\{a_n\}$ *be the sequence, s.t.* $a_n = \frac{1}{n}$ *and n is a natural number. It is easy to see that the mappings f and g are compatible as* $\lim_{n\to\infty} G(fga_n, gfa_n, gfa_n) = 0$ *here* $a_n = \frac{1}{n}$ *s.t.* $\lim_{n\to\infty} fa_n = \lim_{n\to\infty} ga_n = 0$ *for* $0 \in X$.

Now we see some preliminary results of common fixed point theorem as follows. Manoj Kumar Generalized following theorem. Which is stated as

Theorem 2.16 ([7]). *Let(X,G) be complete G-metric space. Let S and T be self mappings on X satisfying following conditions.*

(1). $S(X) \subseteq T(X)$,

(2). S or T is continuous,

(3). $G(Sa,Sb,Sc) \leq qG(Ta,Tb,Tc)$ *for every a,b,c in X and* $0 \leq q < 1$. *And if S and T are Compatible then S and T have Unique common fixed points in X.*

3. Main Result

Now we prove our main result for the pair of self maps of two compatible maps which satisfies different contractive condition as follows.

Theorem 3.1. *Let X be a complete G-metric space.* $S,T : X \to X$ *be two compatible maps on X and which satisfies the following conditions,*

(1). $S(X) \subseteq T(X)$,

(2). S or T is G-continuous,

(3). $G(Sa,Sb,Sc) \leq \alpha G(Sa,Tb,Tc)+\beta G(Ta,Sb,Tc)+\gamma G(Ta,Tb,Sc)+\delta G(Sa,Tb,Tc)$ *for every a,b,c in X and* $\alpha,\beta,\gamma,\delta \geq 0$ *with* $0 \leq \alpha+3\beta+3\gamma+3\delta < 1$.

Then S and T have unique common fixed point in X.

Proof. Let a_0 be an arbitrary element in X by $S(X) \subseteq T(X)$, we construct a sequence $\{b_n\}$ in X s.t. for any a_1 in X $Sa_0 = Ta_1$.In general we take a_{n+1} s.t. $b_n = Sa_n = Ta_{n+1}$, $n = 0,1,2,\ldots$ from given (3) in hypothesis, we have

$$G(Sa_n,Sa_{n+1},Sa_{n+1}) \leq \alpha G(Sa_n,Ta_{n+1},Ta_{n+1})+\beta G(Ta_n,Sa_{n+1},Ta_{n+1})\gamma G(Ta_n,Ta_{n+1},Sa_{n+1})+\delta G(Sa_n,Ta_{n+1},Ta_{n+1})$$

by construction of sequence, we have

$$G(Sa_n,Sa_{n+1},Sa_{n+1}) \leq \alpha G(Sa_n,Sa_n,Sa_n)+\beta G(Sa_{n-1},Sa_{n+1},Sa_n)+\gamma G(Sa_{n-1},Sa_n,Sa_{n+1})+\delta G(Sa_n,Sa_n,Sa_n)$$
$$G(Sa_n,Sa_{n+1},Sa_{n+1}) \leq \beta G(Sa_{n-1},Sa_{n+1},Sa_n)+\gamma G(Sa_{n-1},Sa_n,Sa_{n+1})$$

since by symmetry in variables, we have.

$$G(Sa_n,Sa_{n+1},Sa_{n+1}) \leq (\beta+\gamma)G(Sa_{n-1},Sa_n,Sa_{n+1}) \tag{1}$$

By using Definition (2.1) (5), we have

$$G(Sa_{n-1},Sa_n,Sa_{n+1}) \leq G(Sa_{n-1},Sa_n,Sa_n)+G(Sa_n,Sa_n,Sa_{n+1}) \leq G(Sa_{n-1},Sa_n,Sa_n)+2G(Sa_n,Sa_{n+1},Sa_{n+1})$$

(since by using Proposition 2.12 (3)) from given inequality (3) we have

$$(1-2\beta-2\gamma)G(Sa_n,Sa_{n+1},Sa_{n+1}) \leq (\beta+\gamma)G(Sa_{n-1},Sa_n,Sa_n)$$
$$G(Sa_n,Sa_{n+1},Sa_{n+1}) \leq \frac{(\beta+\gamma)}{(1-2\beta-2\gamma)}G(Sa_{n-1},Sa_n,Sa_n)$$
$$G(Sa_n,Sa_{n+1},Sa_{n+1}) \leq qG(Sa_{n-1},Sa_n,Sa_n)$$

where

$$q = \frac{(\beta+\gamma)}{(1-2\beta-2\gamma)} < 1$$

continuing in this way, we have

$$G(Sa_n,Sa_{n+1},Sa_{n+1}) \leq q^n G(Sa_0,Sa_1,Sa_1) \tag{2}$$

$\therefore$ for all n,m ϵN let $m > n$, by using rectangular inequality we have, now consider,

$$G(b_n,b_m,b_m) \leq G(b_n,b_{n+1},b_{n+1})+G(b_{n+1},b_{n+2},b_{n+2})+\cdots+G(b_{m-1},b_m,b_m)$$
$$G(b_n,b_m,b_m) \leq (q^n+q^{n+1}+\ldots\ldots+q^{m-1})G(b_0,b_1,b_1) \quad \text{(since by using (2))}$$
$$\leq \frac{q^n}{1-q}G(b_0,b_1,b_1)$$

as n,m $\to \infty$, since $q < 1$, $\therefore$ $\frac{q^n}{1-q} \to 0$ as n,m $\to \infty$ therefore R.H.S.of this inequality tends to 0. $\therefore$ we have $\lim_{n\to\infty} G(b_n,b_m,b_m) = 0$ Thus $\{b_n\}$ is a G-cauchy sequence in X. Also as X is a complete G-metric space, $\therefore$ there exists $c \in X$ s.t. $\{b_n\}$ G-converges to c. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} Sa_n = \lim_{n\to\infty} Ta_{n+1} = c$ Given S or T is continuous, Let T is continuous, $\lim_{n\to\infty} TSa_n = \lim_{n\to\infty} TTa_n = Tc$ Also S and T are compatible, $\therefore$ $G(STa_n, TSa_n, TSa_n) = 0$ this gives $\lim_{n\to\infty} STa_n = Tc$ Now from hypothesis (3) we have

$$G(STa_n, Sa_n, Sa_n) \leq \alpha G(STa_n, Ta_n, Ta_n) + \beta G(TTa_n, Sa_n, Ta_n)\gamma G(TTa_n, Ta_n, Sa_n) + \delta G(STa_n, Ta_n, Ta_n)$$

taking lim $as\ n \to \infty$, we have Tc=c Again from (3) we have

$$G(Sa_n, Sc, Sc) \leq \alpha G(Sa_n, Tc, Tc) + \beta G(Ta_n, Sc, Tc)\gamma G(Ta_n, Tc, Sc) + \delta G(Sa_n, Tc, Tc)$$

Taking limit as $n \to \infty$, we have c=Sc. $\therefore$ we have Tc=Sc=c.This shows that c is a common fixed point of S and T.

Uniqueness: If possible c_1 other than c be another common fixed point of S and T.Then $G(c, c_1, c_1) > 0$ and

$$G(c, c_1, c_1) = G(Sc, Sc_1, Sc_1) \leq \alpha G(Sc, Tc_1, Tc_1) + \beta G(Tc, Sc_1, Tc_1)\gamma G(Tc, Tc_1, Sc_1) + \delta G(Sc, Tc_1, Tc_1)$$

$$G(c, c_1, c_1) \leq (\alpha + \beta + \gamma + \delta)G(c, c_1, c_1)$$

$$G(c, c_1, c_1) < G(c, c_1, c_1)$$

This gives contradiction.And which shows that $c = c_1$. This shows that S and T has unique common fixed point in X. □

4. Conclusion

Thus we have proved common fixed point theorem for pair of compatible maps in G-metric space.

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References

[1] S.Banach, *Sur les Operations dans les ensembles abstraits et leur applications aux equaations integrals*, Fund. Math., 3(1922) 133-181 (French).

[2] B.S.Choudhury, *Some fixed point theorems in G-metric spaces*, Nonlinear Science Letters, 1(1)(2012), 25-31.

[3] B.C.Dhage, *Generalized Metric space and mapping with fixed point*, Bull. Calcutta Math Soc., 84(1992).

[4] S.Gahler, *2-Metrices Raume und Ihree Topologische Struktur*, Mathematische Nachrichten, 26(1963), 115-148.

[5] HA.Et al, *Strictly convex and 2-convex 2-normed spaces*, Math. Japonica, 33(3)(1988), 375-384.

[6] G.Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Sci., 9(1986), 771-779.

[7] Manoj Kumar, *Compatible Maps in G-Metric spaces*, Int.Journal of math. Analysis, 6(29)(2012), 1415-1421.

[8] Z.Mustafa and B.Sims, *Some Remarks Concerning D-metric Spaces*, Proceeding of International conference on Fixed point theory and applications, Yokohama publishers, Valenica, 13(19)(2004), 189-198.

[9] Z.Mustafa, *A new approach to Generalized metric Spaces*, J. Nonlinear convex Anal., 7(2)(2006), 289-297.

[10] S.Sesa, *On Weakly Commutativity condition in fixed point consideration*, Publ. Int. Math., 32(46)(1986).